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# Estimation of thermal properties in combined conduction and radiation

H. Y. Li*

*Department of Mechanical Engineering, Hua Fan University, Shihtin, Taipei, Taiwan 22305, Republic of China*

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**Abstract**

An inverse conduction–radiation problem for simultaneous estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function from knowledge of the exit radiation intensities is presented. The inverse problem is solved by using the conjugate gradient method to minimize the error between the calculated exit intensities and the experimental data. The effects of the measurement errors, the conduction-to-radiation parameter, the single scattering albedo, the scattering phase function, and the optical thickness on the accuracy of the inverse analysis are investigated. The results show that the single scattering albedo and the optical thickness can be estimated accurately for exact and noisy data. Estimation of the conduction-to-radiation parameter and the scattering phase function is more difficult than that of the single scattering albedo and the optical thickness because the prediction of the former properties is more sensitive to the measurement errors. 

**Nomenclature**

$a_n$ expansion coefficients for the scattering phase function
**b** $[\omega, \tau_0, N_1, a_1, \ldots, a_{N^*}]^T$
**d** direction of descent
$g_n$ Chandrasekhar polynomials
$I$ radiation intensity
$J$ objective function
$k$ thermal conductivity
$N_1$ conduction-to-radiation parameter
$N^*$ order of the scattering phase function
$\bar{n}$ refractive index
$P_n$ Legendre polynomials
$p$ scattering phase function
$Q^r$ dimensionless radiative heat flux
$q^r$ radiative heat flux
$T$ temperature
$T_1$ temperature at $\tau = 0$
$T_2$ temperature at $\tau = \tau_0$
$Y$ measured dimensionless exit radiation intensities at the surface $\tau = 0$
$y$ coordinate
$Z$ measured dimensionless exit radiation intensities at the surface $\tau = \tau_0$.

*Greek symbols*
$\beta$ extinction coefficient
$\bar{\beta}$ step size
$\gamma$ conjugate coefficient
$\zeta$ random variable
$\theta$ dimensionless temperature
$\theta^*$ dimensionless boundary temperature
$\mu$ direction cosine
$\xi$ eigenvalues
$\sigma$ standard deviation
$\bar{\sigma}$ Stefan–Boltzmann constant
$\tau$ optical coordinate
$\tau_0$ optical thickness
$\psi$ dimensionless radiation intensity
$\omega$ single scattering albedo
$\nabla J$ gradient of the objective function
$\nabla \psi$ sensitivity coefficient vector.

*Superscripts*
$k$ $k$th iteration
$T$ transpose.

* Corresponding author. E-mail: hyli@huafan.hfu.edu.tw

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## 1. Introduction

Inverse problems are important in the field of heat transfer. The inverse analysis provides a great advantage in many engineering applications where direct measurements of the desired quantities are not possible. They have been used extensively to determine crucial parameters in conduction, convection, and radiation. In the inverse heat conduction problems, the surface conditions such as temperature and heat flux or the thermal properties such as thermal conductivity and heat capacity of a material are estimated by utilizing the temperature measurements inside the medium. The problems are known as ill-posed, so that the estimation is very sensitive to the measurement errors of the input data. Various methods have been developed to solve the inverse heat conduction problems, e.g., the function specification method [1], the regularization method [2], the conjugate gradient method with adjoint equation [3], and the mollification method [4]. Several texts have been devoted to this topic of research [5–7]. The inverse radiation problems have also been investigated extensively. They are mainly concerned with the determination of the radiative properties such as the single scattering albedo, the optical thickness, and the scattering phase function [8–13], or the internal temperature profile [14–16] of a medium from the measured radiation data. A comprehensive survey of the inverse radiation problems has been given by McCormick [17–19]. There are many engineering applications where both conduction and radiation are important, e.g., fibrous insulation and glass manufacture. Under such circumstances, the inverse combined conduction and radiation problems are encountered. However, only a limited amount of work is available on this topic. Silva Neto and Ozisik [20] used the Levenberg–Marquardt method to estimate the optical thickness, the single scattering albedo, and the thermal conductivity of a semi-transparent plane-parallel medium. The estimation was based on simulated transmitted exit radiation intensities and interior temperatures of the medium. Manickavasagam and Menguc [21] employed the Levenberg–Marquardt algorithm to estimate the optical thickness and the radiation–conduction parameter of a one-dimensional plane-parallel medium from the input temperature data measured inside the medium. Ruperti et al. [22] estimated the surface temperatures and fluxes from simulated transient temperatures measured inside a one-dimensional semi-transparent slab. A space-marching technique is adopted to solve the problem.

In the present paper, an inverse conduction–radiation problem for simultaneous estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function of a plane-parallel medium from the measured exit radiation intensities is considered. The governing equations for the direct problem will be introduced first. The inverse analysis will then be considered. Test cases will be presented to discuss the effects of the measurement errors, the conduction-to-radiation parameter, the single scattering albedo, the scattering phase function, and the optical thickness on the estimation.

## 2. Direct problem

Consider steady-state combined conduction and radiation heat transfer in a grey, absorbing, emitting, and anisotropic scattering slab of optical thickness $\tau_0$, with transparent boundaries and subjected to isotropic incident radiation at the boundary $\tau = 0$. The boundary surfaces are kept at specified constant temperatures $T_1$ and $T_2$, respectively. The dimensionless form of the energy equation can be expressed as [23]

$$\frac{d^2\theta}{d\tau^2} - \frac{1}{N_1}\frac{dQ^r}{d\tau} = 0 \tag{1a}$$

with the boundary conditions

$$\theta = 1 \quad \text{at } \tau = 0 \tag{1b}$$

$$\theta = \theta^* \quad \text{at } \tau = \tau_0 \tag{1c}$$

where $\theta = T/T_1$ is the dimensionless temperature, $\theta^* = T_2/T_1$ is the dimensionless boundary temperature, $\tau = \beta y$ is the optical coordinate, $N_1 = k\beta/4\bar{n}^2\bar{\sigma}T_1^3$ is the conduction-to-radiation parameter, and $Q^r = q^r/4\bar{n}^2\bar{\sigma}T_1^4$ is the dimensionless radiative heat flux. The derivative of the dimensionless radiative heat flux is determined from the solution of the equation of radiative transfer. The equation of radiative transfer in dimensionless form is given by [23]

$$\mu\frac{\partial\psi(\tau,\mu)}{\partial\tau} + \psi(\tau,\mu) = (1-\omega)\theta^4(\tau) + \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')\psi(\tau,\mu')\,d\mu' \tag{2a}$$

$$\psi(0,\mu) = 1 \quad \mu > 0 \tag{2b}$$

$$\psi(\tau_0,-\mu) = 0 \quad \mu > 0 \tag{2c}$$

where $\psi = \pi I/\bar{n}^2\bar{\sigma}T_1^4$ is the dimensionless radiation intensity, $\mu$ is the cosine of the angle between the $\tau$ coordinate and the direction of the radiation intensity, $\omega$ is the single scattering albedo, and $p(\mu,\mu')$ is the scattering phase function, which is expressed in terms of the Legendre polynomials

$$p(\mu,\mu') = \sum_{n=0}^{N^*} a_n P_n(\mu)P_n(\mu') \quad \text{with } a_0 = 1. \tag{2d}$$

Equations (1) and (2) provide the complete mathematical formulation for the one-dimensional steady-state combined conduction and radiation. The solution of the equation of radiative transfer is obtained by the $P_N$ method. In this approximation, the dimensionless radiation intensity is expressed in the form [24]

$$\psi(\tau,\mu)=\sum_{n=0}^{N}\frac{2n+1}{2}P_n(\mu)\sum_{j=1}^{J^*}\times[A_j\,e^{(-\tau/\xi_j)}+(-1)^n B_j\,e^{(-(\tau_0-\tau)/\xi_j)}]g_n(\xi_j)+\varphi_p(\tau,\mu) \quad (3)$$

where $J^*=(N+1)/2$, $N$ is an odd integer, $g_n(\xi)$ are the Chandrasekhar polynomials determined from the recurrence formula

$$(n+1)g_{n+1}(\xi)=h_n\xi g_n(\xi)-ng_{n-1}(\xi) \quad (4)$$

for $n=0,1,\ldots,N$, with $g_0(\xi)=1$ and $h_n=2n+1-\omega a_n$. The eigenvalues $\xi_j$, $j=1,2,\ldots,J^*$, are the $J^*$ positive solutions of the following eigenvalue problem:

$$\frac{n(n-1)}{h_nh_{n-1}}g_{n-2}(\xi)+\frac{1}{h_n}\left[\frac{(n+1)^2}{h_{n+1}}+\frac{n^2}{h_{n-1}}\right]g_n(\xi)+\frac{(n+2)(n+1)}{h_{n+1}h_n}g_{n+2}(\xi)=\xi^2g_n(\xi) \quad (5)$$

for $n=0,2,4,\ldots,N-1$. $\varphi_p(\tau,\mu)$ is a particular solution of equation (2a) corresponding to the inhomogeneous term $(1-\omega)\theta^4(\tau)$ and it is determined from [25]

$$\varphi_p(\tau,\mu)=\sum_{n=0}^{N}\frac{2n+1}{2}P_n(\mu)\times\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}\left[\int_0^{\tau}(1-\omega)\theta^4(\tau')\,e^{(-(\tau-\tau')/\xi_j)}\,d\tau'+(-1)^n\int_{\tau}^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-(\tau'-\tau)/\xi_j)}\,d\tau'\right]g_n(\xi_j) \quad (6)$$

where

$$c_j=\left[\sum_{n=1}^{J^*}g^2_{2n-2}(\xi_j)h_{2n-2}\right]^{-1}. \quad (7)$$

The constants $A_j$ and $B_j$ are determined by requiring the solution given by equation (3) to satisfy the Marshak boundary conditions

$$\sum_{n=0}^{N}\frac{2n+1}{2}S_{\alpha,n}\sum_{j=1}^{J^*}[A_j+(-1)^nB_j\,e^{(-\tau_0/\xi_j)}]g_n(\xi_j)=S_{\alpha,o}-\sum_{n=0}^{N}(-1)^n\frac{2n+1}{2}S_{\alpha,n}\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\times\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-\tau'/\xi_j)}\,d\tau' \quad (8)$$

and

$$\sum_{n=0}^{N}\frac{2n+1}{2}S_{\alpha,n}\sum_{j=1}^{J^*}[(-1)^nA_j\,e^{(-\tau_0/\xi_j)}+B_j]g_n(\xi_j)=-\sum_{n=0}^{N}(-1)^n\frac{2n+1}{2}S_{\alpha,n}\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\times\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-(\tau_0-\tau')/\xi_j)}\,d\tau' \quad (9)$$

for $\alpha=0,1,\ldots,(N-1)/2$, where

$$S_{\alpha,n}=\int_0^1P_{2\alpha+1}(\mu)P_n(\mu)\,d\mu. \quad (10)$$

Once $A_j$ and $B_j$ are available, the dimensionless exit radiation intensities at $\tau=0$ and $\tau=\tau_0$, and the derivative of the dimensionless radiative heat flux are calculated from [24]

$$\psi(0,-\mu)=\sum_{n=0}^{N}(-1)^n\frac{2n+1}{2}P_n(-\mu)\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\times\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-\tau'/\xi_j)}\,d\tau'-e^{(-\tau_0/\mu)}\sum_{n=0}^{N}\frac{2n+1}{2}\times P_n(-\mu)\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-(\tau_0-\tau')/\xi_j)}\,d\tau'+\frac{\omega}{2}\sum_{n=0}^{N}a_nP_n(\mu)\sum_{j=1}^{J^*}\xi_j\left[(-1)^nA_j\frac{1-e^{(-\tau_0/\mu)}\,e^{(-\tau_0/\xi_j)}}{\mu+\xi_j}+B_j\frac{e^{(-\tau_0/\mu)}-e^{(-\tau_0/\xi_j)}}{\mu-\xi_j}\right]g_n(\xi_j)\quad \mu>0 \quad (11)$$

$$\psi(\tau_0,\mu)=e^{(-\tau_0/\mu)}+\sum_{n=0}^{N}\frac{2n+1}{2}P_n(\mu)\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\times\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-(\tau_0-\tau')/\xi_j)}\,d\tau'-e^{(-\tau_0/\mu)}\sum_{n=0}^{N}(-1)^n\frac{2n+1}{2}P_n(\mu)\sum_{j=1}^{J^*}\frac{c_j}{\xi_j}g_n(\xi_j)\times\int_0^{\tau_0}(1-\omega)\theta^4(\tau')\,e^{(-\tau'/\xi_j)}\,d\tau'+\frac{\omega}{2}\sum_{n=0}^{N}a_nP_n(\mu)\sum_{j=1}^{J^*}\times\xi_j\left[A_j\frac{e^{(-\tau_0/\mu)}-e^{(-\tau_0/\xi_j)}}{\mu-\xi_j}+(-1)^nB_j\frac{1-e^{(-\tau_0/\mu)}\,e^{(-\tau_0/\xi_j)}}{\mu+\xi_j}\right]g_n(\xi_j)\quad \mu>0 \quad (12)$$

and

$$\frac{dQ^r}{d\tau}=(1-\omega)\left\{\theta^4(\tau)-\frac{1}{2}\sum_{j=1}^{J^*}(A_j\,e^{(-\tau/\xi_j)}+B_j\,e^{(-(\tau_0-\tau)/\xi_j)})\right.$$

$$-\frac{(1-\omega)}{2}\frac{c_j}{\xi_j}\left[\int_0^{\tau}\theta^4(\tau')\,e^{(-(\tau-\tau')/\xi_j)}\,d\tau' + \int_{\tau}^{\tau_0}\theta^4(\tau')\,e^{(-(\tau'-\tau)/\xi_j)}\,d\tau'\right]\Bigg\}. \quad (13)$$

The solution of the energy equation is determined by the finite difference method. The physical domain is divided into $M^*+1$ points with mesh size $\Delta\tau = \tau_0/M^*$. The finite difference form of the energy equation is

$$\frac{\theta_{i-1}-2\theta_i+\theta_{i+1}}{(\Delta\tau)^2}-\frac{1}{N_1}\left(\frac{dQ^r}{d\tau}\right)_i = 0 \quad i = 1,2,\ldots,M^*-1 \quad (14a)$$

with the boundary conditions

$$\theta_0 = 1 \quad (14b)$$

$$\theta_{M^*} = \theta^*. \quad (14c)$$

An iterative process is needed in solving the energy equation and the equation of radiative transfer. A temperature profile is first assumed, and the equation of radiative transfer is solved for the radiation intensity and the derivative of the radiative heat flux. The energy equation is solved next for a new temperature profile. These two temperature profiles are compared. The process is continued until a specified convergent criterion is achieved.

## 3. Inverse problem

In the direct problem, the thermal properties $\omega$, $\tau_0$, $N_1$, and $a_n$ are given to determine the temperature distribution and the radiation intensity. In the inverse problem, the exit radiation intensities are assumed to be measured and the parameters $\omega$, $\tau_0$, $N_1$, and $a_n$ are recovered by using the measured data. The estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function from the knowledge of the exit radiation intensities measured at different directions can be constructed as a problem of minimization of the objective function

$$J = \sum_{i=1}^{M_1}[\psi(0,-\mu_i;\mathbf{b})-Y(-\mu_i)]^2+\sum_{i=1}^{M_2}[\psi(\tau_0,\mu_i;\mathbf{b})-Z(\mu_i)]^2 \quad (15)$$

where $\psi(0,-\mu_i;\mathbf{b})$ and $\psi(\tau_0,\mu_i;\mathbf{b})$ are the calculated dimensionless exit radiation intensities at $\tau = 0$ and $\tau = \tau_0$, respectively, for an estimated vector $\mathbf{b} = [\omega,\tau_0,N_1,a_1,\ldots,a_{N^*}]^T$; $Y(-\mu_u)$ and $Z(\mu_i)$ are the measured dimensionless exit radiation intensities at $\tau = 0$ and $\tau = \tau_0$, respectively.

In this study, the conjugate gradient method is employed to solve the inverse conduction–radiation problem. The iterative process is [26]

$$\mathbf{b}^{k+1} = \mathbf{b}^k - \bar{\beta}^k\mathbf{d}^k \quad (16)$$

where $\bar{\beta}^k$ is the step size, $\mathbf{d}^k$ is the direction of descent which is determined from

$$\mathbf{d}^k = \nabla J^T(\mathbf{b}^k)+\gamma^k\mathbf{d}^{k-1} \quad (17)$$

and the conjugate coefficient $\gamma^k$ is computed from

$$\gamma^k = \frac{\nabla J(\mathbf{b}^k)\nabla J^T(\mathbf{b}^k)}{\nabla J(\mathbf{b}^{k-1})\nabla J^T(\mathbf{b}^{k-1})} \quad \text{with } \gamma^0 = 0. \quad (18)$$

Here the row vector

$$\nabla J = \left[\frac{\partial J}{\partial\omega},\frac{\partial J}{\partial\tau_0},\frac{\partial J}{\partial N_1},\frac{\partial J}{\partial a_1},\ldots,\frac{\partial J}{\partial a_{N^*}}\right] \quad (19)$$

is the gradient of the objective function. The step size is determined from

$$\bar{\beta}^k = \left\{\sum_{i=1}^{M_1}[\psi(0,-\mu_i;\mathbf{b}^k)-Y(-\mu_i)]\nabla\psi(0,-\mu_i;\mathbf{b}^k)\mathbf{d}^k + \sum_{i=1}^{M_2}[\psi(\tau_0,\mu_i;\mathbf{b}^k)-Z(\mu_i)]\nabla\psi(\tau_0,\mu_i;\mathbf{b}^k)\mathbf{d}^k\right\} \Bigg/ \left\{\sum_{i=1}^{M_1}[\nabla\psi(0,-\mu_i;\mathbf{b}^k)\mathbf{d}^k]^2+\sum_{i=1}^{M_2}[\nabla\psi(\tau_0,\mu_i;\mathbf{b}^k)\mathbf{d}^k]^2\right\} \quad (20)$$

where $\nabla\psi$ is the sensitivity coefficient vector

$$\nabla\psi = \left[\frac{\partial\psi}{\partial\omega},\frac{\partial\psi}{\partial\tau_0},\frac{\partial\psi}{\partial N_1},\frac{\partial\psi}{\partial a_1},\ldots,\frac{\partial\psi}{\partial a_{N^*}}\right]. \quad (21)$$

The component of the sensitivity coefficient vector defined as the first derivative of the dimensionless radiation intensity with respect to the unknown parameter is determined by the finite difference method

$$\frac{\partial\psi}{\partial b_j} \cong \frac{\psi(b_0,b_1,\ldots,b_j+\Delta b_j,\ldots,b_M)}{\Delta b_j} \quad (22)$$

for $j = 0,1,\ldots,M$. A large component of the sensitivity coefficient vector indicates that the dimensionless radiation intensity is sensitive to changes in that parameter, while a small component implies that the dimensionless radiation intensity is insensitive to changes in that parameter. The gradient of the objective function is determined by differentiating equation (15) with respect to $b_j$ to obtain

$$\frac{\partial J}{\partial b_j} = 2\sum_{i=1}^{M_1}[\psi(0,-\mu_i;\mathbf{b})-Y(-\mu_i)]\frac{\partial\psi(0,-\mu_i;\mathbf{b})}{\partial b_j}$$

$$+2\sum_{i=1}^{M_2}[\psi(\tau_0,\mu_i;\mathbf{b})-Z(\mu_i)]\frac{\partial\psi(\tau_0,\mu_i;\mathbf{b})}{\partial b_j} \quad (23)$$

for $j = 0, 1, \ldots, M$.

If the problem contains no measurement errors, the condition

$$J(\mathbf{b}^{k+1}) < \delta^* \quad (24)$$

can be used for terminating the iterative process, where $\delta^*$ is a small specified positive number. However, the measured radiation intensities contain measurement errors. Following the computational experience, we use the discrepancy principle [27]

$$J(\mathbf{b}^{k+1}) < (M_1+M_2)\sigma^2 \quad (25)$$

as the stopping criterion, where $\sigma$ is the standard deviation of the measurement errors.

The computational procedure for the solution of the inverse conduction–radiation problem can be summarized as follows:

Step 1: Pick an initial guess $\mathbf{b}^0$. Set $k = 0$.
Step 2: Solve the direct problem to compute the dimensionless exit radiation intensities $\psi(0,-\mu_i;\mathbf{b}^k)$ and $\psi(\tau_0,\mu_i;\mathbf{b}^k)$.
Step 3: Calculate the objective function. Terminate the iteration process if the specified stopping criterion is satisfied. Otherwise go to Step 4.
Step 4: Compute the sensitivity coefficient vector $\nabla\psi$.
Step 5: Knowing $\nabla\psi$, $\psi(0,-\mu_i;\mathbf{b}^k)$, $\psi(\tau_0,\mu_i;\mathbf{b}^k)$, $Y(-\mu_i)$, and $Z(\mu_i)$, compute the gradient of the objective function $\nabla J(\mathbf{b}^k)$.
Step 6: Knowing $\nabla J(\mathbf{b}^k)$, compute the conjugate coefficient $\gamma^k$ and the direction of descent $\mathbf{d}^k$.
Step 7: Knowing $\nabla\psi$, $\psi(0,-\mu_i;\mathbf{b}^k)$, $\psi(\tau_0,\mu_i;\mathbf{b}^k)$, $Y(-\mu_i)$, $Z(\mu_i)$, and $\mathbf{d}^k$, compute the step size $\bar{\beta}^k$.
Step 8: Knowing $\bar{\beta}^k$ and $\mathbf{d}^k$, compute $\mathbf{b}^{k+1}$. Set $k = 0$ if $k = M+1$ or $k = k+1$ if $k \neq M+1$ and go to Step 2.

## 4. Results and discussion

Several test cases are presented to demonstrate the proposed inverse algorithm for simultaneously estimating the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function from the knowledge of the exit radiation intensities. The effects of the measurement errors, the conduction-to-radiation parameter, the single scattering albedo, the scattering phase function, and the optical thickness on the results of the inverse analysis are investigated. In order to simulate the measured exit intensities with measured errors, $Y$ and $Z$, random errors of standard deviation $\sigma$ are added to the exact intensities computed from the solution of the direct problem. Thus, we have

$$Y_{\text{measured}} = Y_{\text{exact}}+\sigma\zeta \quad (26)$$

and

$$Z_{\text{measured}} = Z_{\text{exact}}+\sigma\zeta \quad (27)$$

where $\zeta$ is a random variable with normal distribution, zero mean and unit standard deviation. The exit radiation intensities are measured at the surfaces $\tau = 0$ and $\tau = \tau_0$, and 20 measurement points are taken at each surface over the polar angle interval $0 \leqslant \theta \leqslant \pi/2$ for all the cases considered here. The data are used as input to reconstruct the unknown properties from the inverse problem where the dimensionless boundary temperature $\theta^*$ is taken as 0.1.

In the first case, the single scattering albedo, the optical thickness, and the conduction-to-radiation parameter are assumed to be 0.5, 1, 1, respectively. Phase function I [28] of Table 1 is used for the scattering characteristics of the medium. The results of the inverse analysis for both exact and noisy input data are shown in Fig. 1. The estimation of the inverse problem is excellent for exact input data, i.e., $\sigma = 0$. The accuracy of the inverse analysis is also good for simulated experimental data containing errors of standard deviation $\sigma = 0.002$ and $\sigma = 0.004$. Increasing $\sigma$ from 0.002 to 0.004, the accuracy of the estimation decreases. It is noted that the estimation for the single scattering albedo and the optical thickness is less sensitive to the measurement errors, while the estimation for the conduction-to-radiation parameter and the scattering phase function is more sensitive to the measurement errors.

Figure 2 is presented to show the effects of the conduction-to-radiation parameter on the accuracy of the inverse analysis. The thermal properties used are the same as those for Fig. 1 except in this case $N_1 = 0.1$. The agreements between the estimated and the exact values of the single scattering albedo and the optical thickness

Table 1
The expansion coefficients for the scattering phase functions

| | Phase function I | Phase function II |
|---|---|---|
| $n$ | $a_n$ | $a_n$ |
| 0 | 1 | 1 |
| 1 | −1.2 | 0.84664 |
| 2 | 0.5 | 0.03635 |
| 3 | | −0.04477 |
| 4 | | 0.33367 |
| 5 | | 0.13727 |
| 6 | | 0.02852 |
| 7 | | 0.00353 |
| 8 | | 0.00027 |

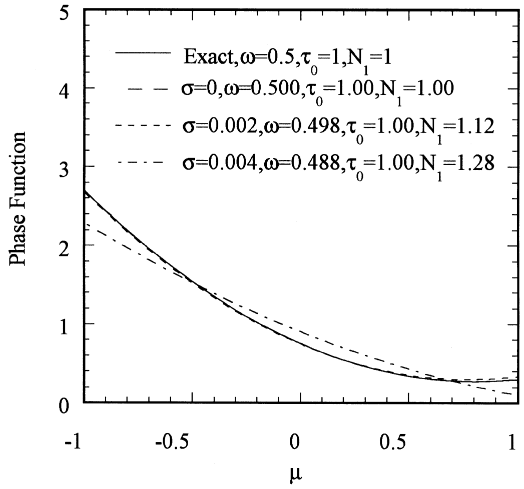


Fig. 1. Estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function for $\omega = 0.5$, $\tau_0 = 1$, $N_1 = 1$, phase function I, $\theta^* = 0.1$.

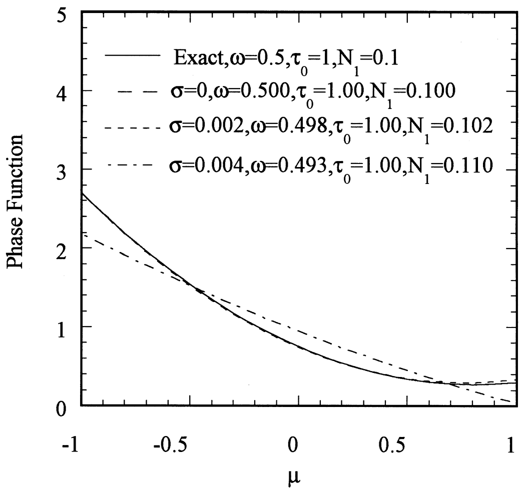


Fig. 2. Estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function for $\omega = 0.5$, $\tau_0 = 1$, $N_1 = 0.1$, phase function I, $\theta^* = 0.1$.

are very good. The estimation of the conduction-to-radiation parameter and the scattering phase function is more difficult than that of the single scattering albedo and the optical thickness. It is due to the fact that the components of the sensitivity coefficient vector for the single scattering albedo and the optical thickness are much larger than those for the conduction-to-radiation parameter and the scattering phase function.

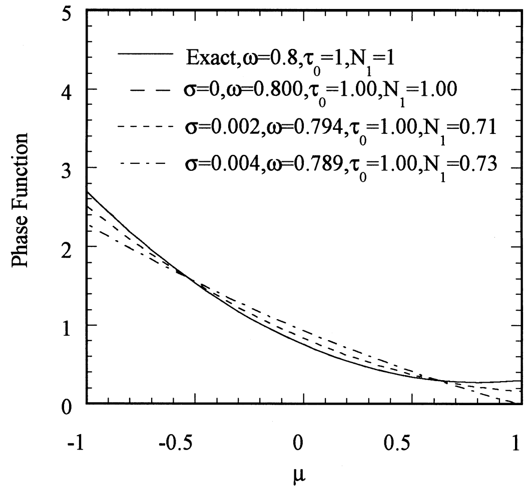


Fig. 3. Estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function for $\omega = 0.8$, $\tau_0 = 1$, $N_1 = 1$, phase function I, $\theta^* = 0.1$.

Figures 3 and 4 are intended to demonstrate the effects of the single scattering albedo on the estimation. The values of the single scattering albedo are 0.8 and 0.2 for Figs 3 and 4, respectively, and other properties are the same as those for Fig. 1. The prediction of the properties is acceptable. Again, the estimations for the conduction-to-radiation parameter and the scattering phase function are more sensitive to the measurement errors.

Figure 5 shows the results of the inverse analysis for a medium with single scattering albedo 0.5, optical thickness 1, conduction-to-radiation parameter 1, and phase

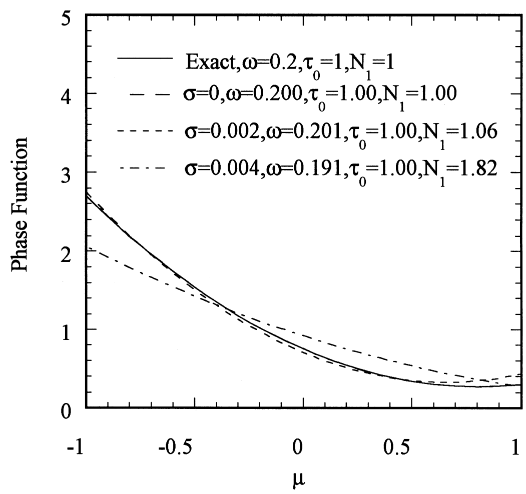


Fig. 4. Estimation of the single scattering albedo, the optical thickness, the conduction-to-parameter, and the scattering phase function for $\omega = 0.2$, $\tau_0 = 1$, $N_1 = 1$, phase function I, $\theta^* = 0.1$.

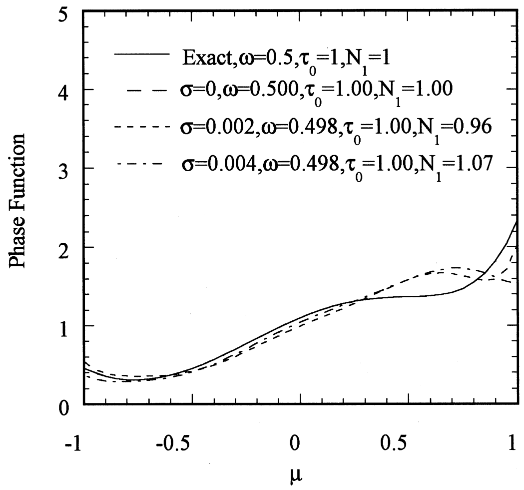


Fig. 5. Estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function for $\omega = 0.5$, $\tau_0 = 1$, $N_1 = 1$, phase function II, $\theta^* = 0.1$.

function II [23] of Table 1. The accuracy of the estimation for simulated experimental data with $\sigma = 0$, $\sigma = 0.002$ and $\sigma = 0.004$ is good. Comparing Fig. 5 with Fig. 1, it is noted that the estimated results are satisfactory for both forward scattering and backward scattering media.

The effects of the optical thickness on the inverse analysis are shown in Fig. 6. The properties used are the same as those for Fig. 1 except in this case $\tau_0 = 5$. The accuracy of the estimation for the single scattering albedo and the optical thickness is very good. The estimation of the conduction-to-radiation parameter and the scattering phase function is more difficult than that of the single scattering albedo and the optical thickness because the prediction of the former properties is more sensitive to the measurement errors.

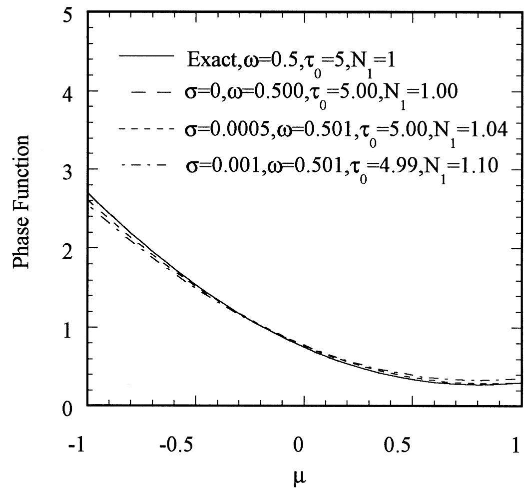


Fig. 6. Estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function for $\omega = 0.5$, $\tau_0 = 5$, $N_1 = 1$, phase function I, $\theta^* = 0.1$.

## 5. Conclusions

The inverse conduction–radiation problem for simultaneous estimation of the single scattering albedo, the optical thickness, the conduction-to-radiation parameter, and the scattering phase function from the knowledge of the exit radiation intensities has been considered. The conjugate gradient method is adopted to solve the problem. Both exact and noisy data have been used to demonstrate the inverse algorithm. The results show that the single scattering albedo and the optical thickness can be estimated accurately for both exact and noisy data. The estimation of the conduction-to-radiation parameter and the scattering phase function is more difficult and is more sensitive to the measurement errors. The inverse method developed in this paper can be extended for non-grey, inhomogeneous radiative heat transfer models and general boundary conditions.

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## References

[1] J.V. Beck, B. Blackwell, A. Haji-Sheikh, Comparison of some inverse heat conduction methods using experimental data, International Journal of Heat and Mass Transfer 39 (1996) 3649–3657.
[2] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill-Posed Problems, Winston, Washington, DC, 1977.
[3] Y. Jarny, M.N. Ozisik, J.P. Bardon, A general optimization method using adjoint equation for solving multi-dimensional inverse heat conduction, International Journal of Heat and Mass Transfer 34 (1991) 2911–2919.
[4] D.A. Murio, The Mollification Method and the Numerical Solutions of Ill-Posed Problems, Wiley, New York, 1993.
[5] J.V. Beck, B. Blackwell, C.R. St Clair Jr., Inverse Heat conduction: Ill-Posed Problems, Wiley, New York, 1985.
[6] E. Hensel, Inverse Theory and Applications for Engineers, Prentice-Hall, New Jersey, 1991.
[7] O.M. Alifanov, Inverse Heat Transfer Problems, Springer-Verlag, New York, 1994.
[8] C.E. Siewert, A new approach to the inverse problem, Journal of Mathematical Physics 19 (1978) 2619–2621.

[9] W.L. Dunn, Inverse Monte Carlo solutions for radiative transfer in inhomogeneous media, Journal of Quantitative Spectroscopy and Radiative Transfer 29 (1983) 19–26.

[10] C.H. Ho, M.N. Ozisik, An inverse radiation problem, International Journal of Heat and Mass Transfer 32 (1989) 335–341.

[11] S. Subramaniam and M.P. Menguc, Solution of the inverse radiation problem for inhomogeneous and anisotropically scattering media using a Monte Carlo technique, International Journal of Heat and Mass Transfer 34 (1991) 253–266.

[12] K. Kamiuto, An iterative method for inverse scattering problems, Journal of Quantitative Spectroscopy and Radiative Transfer 49 (1993) 1–13.

[13] H.Y. Li, C.Y. Yang, A genetic algorithm for inverse radiation problems, International Journal of Heat and Mass Transfer 40 (1997) 1545–1549.

[14] H.Y. Li, M.N. Ozisik, Identification of the temperature profile in an absorbing, emitting, and isotropically scattering medium by inverse analysis, Journal of Heat Transfer 114 (1992) 1060–1063.

[15] C.E. Siewert, A radiative-transfer inverse-source problem for a sphere, Journal of Quantitative Spectroscopy and Radiative Transfer 52 (1994) 157–160.

[16] H.Y. Li, Inverse radiation problem in two-dimensional rectangular media, Journal of Thermophysics and Heat Transfer 11 (1997) 556–561.

[17] N.J. McCormick, Recent developments in inverse scattering transport methods, Transport Theory and Statistical Physics 13 (1984) 15–28.

[18] N.J. McCormick, Methods for solving inverse problems for radiation transport—an update, Transport Theory and Statistical Physics 15 (1986) 759–772.

[19] N.J. McCormick, Inverse radiative transfer problems: a review, Nuclear Science and Engineering 112 (1992) 185–198.

[20] A.J. Silva Neto, M.N. Ozisik, An inverse problem of estimating thermal conductivity, optical thickness, and single scattering albedo of a semi-transparent medium, The First Conference in a Series on Inverse Problems in Engineering: Theory and Practice, Palm Coast, Florida, 1993, pp. 267–273.

[21] S. Manickavasagam, M.P. Menguc, Inverse radiation/conduction problem in plane-parallel media, Radiative Heat Transfer: Theory and Applications, 1993, ASME HTD—Vol. 244, pp. 67–75.

[22] N.J. Ruperti Jr, M. Raynaud and J.F. Sacadura, A method for the solution of the coupled inverse heat conduction–radiation problem, Journal of Heat Transfer 118 (1996) 10–17.

[23] M.N. Ozisik, Radiative Transfer, Wiley, New York, 1973.

[24] M. Benassi, M. Cotta, C.E. Siewert, The $P_N$ method for radiative transfer problems with reflective boundary conditions, Journal of Quantitative Spectroscopy and Radiative Transfer 30 (1983) 547–553.

[25] C.E. Siewert, J.R. Thomas, A particular solution for the $P_N$ method in radiative transfer, Journal of Quantitative Spectroscopy and Radiative Transfer 43 (1990) 433–436.

[26] M.R. Hestenes, Conjugate Direction Methods in Optimization, Springer-Verlag, New York, 1980.

[27] O.M. Alifanov, Solution of an inverse problem of heat conduction by iteration methods, Journal of Engineering Physics 26 (1974) 471–476.

[28] G. Parthasarathy, S.V. Patankar, J.C. Chai, H.S. Lee, Monte Carlo solutions for radiative heat transfer in irregular two-dimensional geometries, Radiative Heat Transfer: Current Research, 1994, ASME HTD-Vol. 276, pp. 191–201.